CSCE 689: Advanced Graph Algorithms

Homework 1

Due date: Sept 10 *Name:*

Problem 0. Remember to write in your name. Who did you work with on this homework, if anyone? Remember to include citations to any sources you use in the homework.

Problem 1. Linear algebra review (5 points).

a) Let $\mathbf{A}$ be an $n \times n$ matrix and $\mathbf{b}$ be a length n vector. If t is the exact number of solutions for the linear system $\mathbf{Ax} = \mathbf{b}$, what are the possible values for t?

b) Compute the eigenvalues for the following matrix, and an eigenvector corresponding to the smallest one. Show how to compute them by hand without using a computer or calculator.

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 1 & 1 \end{bmatrix}$$

c) What is the geometric multiplicity of an eigenvalue? What is the algebraic multiplicity of an eigenvalue? Given an example of a 2×2 matrix where the geometric and algebraic multiplicities of the eigenvalue(s) are equal. Given another example where they are different.

d) Assume that the matrix $\mathbf{A}$ has at least one eigenvalue of zero. Is it possible for $\mathbf{A}^{-1}$ to exist? Explain why or why not.

Problem 2. This problem will walk through a proof of the following spectral theorem.

Spectral Theorem. If $\mathbf{A}$ is a real-valued symmetric matrix, all of its eigenvalues are real, and eigenvectors corresponding to distinct eigenvalues are orthogonal.

To prove this, we first review complex numbers, vectors, and matrices. The complex conjugate of a complex number $z = a + bi \in \mathbb{C}$ is given by $\overline{z} = a - bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. For a vector $\mathbf{x} \in \mathbb{C}^n$ and a matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$, complex conjugates are defined entrywise, i.e., $\overline{\mathbf{x}}(i) = \overline{\mathbf{x}(i)}$ and $\overline{\mathbf{A}(i,j)} = \overline{\mathbf{A}}(i,j)$. Consider the following two facts:

Fact 1: For any vector $\mathbf{x} \in \mathbb{C}^n$, the product $\overline{\mathbf{x}}^T\mathbf{x}$ is real and nonnegative, and is only zero if $\mathbf{x} = \mathbf{0}$.

Fact 2: For $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\overline{\mathbf{A}\mathbf{x}} = \overline{\mathbf{A}}\overline{\mathbf{x}}$.

a) (2 points) Prove Fact 1.

b) (2 points) Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ be two complex numbers. Show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and that $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$ (this is the key step to showing why Fact 2 is true. You do not need to provide a full prove of Fact 2).

c) (5 points) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a real symmetric matrix and λ be an eigenvalue. Use Facts 1 and 2 to prove that λ is real. Carefully justify all of your steps.

d) (4 points) Let $\mathbf{A}$ be a real symmetric matrix. Assume $\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ and $\mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2$. Prove that if $\lambda_1 \neq \lambda_2$, then $\mathbf{x}_1^T\mathbf{x}_2 = 0$.

e) (2 points) A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is:

positive definite if for every nonzero $\mathbf{x} \in \mathbb{R}^n$ it satisfies $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$.

positive semidefinite if for all $\mathbf{x} \in \mathbb{R}^n$ it satisfies $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$.

If $\mathbf{A}$ is positive semidefinite, what can we say about its eigenvalues? What can we say about its eigenvalues if it is positive definite? Explain why in each case.

Problem 3. (10 points) Let $\mathbf{L}$ be the Laplacian matrix for a simple unweighted and undirected graph $G = (V, E)$. Recall that the jth entry of the ith column is given by

$$[\mathbf{L}\mathbf{e}_i]_j = \begin{cases} d_i & \text{if } j = i \\ -1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathbf{e}_i$ is the indicator vector for the ith node: it is all zeros except for a 1 in the ith position.

a) Let $\mathbf{B}$ be the incidence matrix obtained by assigning an arbitrary direction to the edges of the graph G. This means that the row-i, column-k entry of $\mathbf{B}$ is

$$B_{i,k} = \begin{cases} 1 & \text{if } i \text{ is the head of the } k\text{th edge} \\ -1 & \text{if } i \text{ is the tail of the } k\text{th edge} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Prove that $\mathbf{L} = \mathbf{B}\mathbf{B}^T$, regardless of the direction assigned to edges. (Hint: for a fixed i, how do we compute the i,j entry of $\mathbf{B}\mathbf{B}^T$, and what is it when $i = j$ or $i \neq j$?)

b) Is $\mathbf{L}$ positive definite? Is it positive semidefinite? Prove your answer in each case.

c) If G is disconnected, prove that the smallest eigenvalue of $\mathbf{L}$ has geometric multiplicity at least two. Hint: first, what is the smallest eigenvalue? Then, recall the definition of geometric multiplicity, and think about how to construct orthogonal eigenvectors for the smallest eigenvalue.

Problem 4. Let $\mathbf{A}$ be the adjacency matrix for a simple unweighted and undirected graph $G = (V, E)$, and $\mathbf{L}$ be the Laplacian matrix.

a) (3 points) Let $S \subseteq V$ be a subset of nodes, and define a vector $\mathbf{e}_S$ so that

$$[\mathbf{e}_S]_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$

Prove that $\mathbf{e}_S^T \mathbf{L} \mathbf{e}_S = \mathbf{cut}(S)$. You can use the fact that for a general $n \times n$ matrix $\mathbf{M}$ and vector $\mathbf{x}$,

$$\mathbf{x}^T \mathbf{M} \mathbf{x} = \sum_{i=1}^{n} \sum_{i=1}^{n} M_{ij} x_i x_j$$

b) (5 points) Prove by induction that $[\mathbf{A}^k]_{ij}$ is the number of paths of length k from node i to node j.

c) (2 points) Prove that $\text{trace}(\mathbf{A}^3)/6$ equals the number of triangles in the graph.

Problem 5. Coding assignment (0 points, optional!). The power method is a simple iterative strategy that can be used to find the largest magnitude eigenvalue of a symmetric[1] matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ that applies the following strategy

1. Select a random $\mathbf{x}^{(0)} \in \mathbb{R}^n$.
2. For $i = 1, 2, \ldots$

$$\mathbf{x}^{(i)} = \frac{\mathbf{A}\mathbf{x}^{(i-1)}}{\|\mathbf{A}\mathbf{x}^{(i-1)}\|_2}$$

Theorem. Assume $\mathbf{A}$ is symmetric and has eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfying $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$. If $\mathbf{x}^{(0)}$ is chosen so that $\mathbf{x}_1^T \mathbf{x}^{(0)} > 0$ for some eigenvector $\mathbf{x}_1$ corresponding to the eigenvalue λ_1, then $\mathbf{x}^{(i)}$ converges to an eigenvector $\mathbf{x}$ where $\mathbf{A}\mathbf{x} = \lambda_1 \mathbf{x}$.

For the coding assignment, you will implement the power method in Julia. On Canvas, you will find files `graph-mat.mtx`, `power-method.jl`, and `test-power-method.jl`. You will fill in your code for the power method in `power-method.jl`, and test it in `test-power-method.jl`.

You may also choose to implement the method in a different programming language, though there is no starter template for another language.

Since this homework is optional, you will not upload anything. An answer will be provided in case you would like to self-check your work.

[1] The symmetry assumption can be relaxed, but here we will just focus on the case where $\mathbf{A}$ is symmetric.