CSCE 689: Advanced Graph Algorithms

Lecture 1: Course Intro, Graph Basics

Date: August 25, 2022 *Lecturer: Nate Veldt*

1 Graph Basics, Max-flow and Min-cuts

1.1 Graph Notation and Terminology

An (undirected) graph $G = (V, E)$ is defined by

An edge between nodes i and j is denoted by ____________________

We can also denote an edge by ____________________

If $(i, j) \in E$, we say i and j are _______________. The neighborhood of node i is the set of nodes adjacent to it:

The *degree* of i is the number of neighbors it has: __________

1.2 Basic graph classes

Simple graphs:

Two important generalizations

- Weighted:

- Directed:

Other simple types of graphs

1.3 Basic edge structures

- **Triangle**: set of three nodes that all share edges (also a size-3 cycle):

$$\{i, j, k\} \subseteq V \text{ such that } \{(i, j), (i, k), (j, k)\} \in E$$

- **Path**: is a sequence of edges joining a sequence of vertices:

$$\{i_1, i_2, \dots i_k\} \subseteq V \text{ where } (i_1, i_2) \in E, (i_2, i_3) \in E, \dots, (i_{k-1}, i_k) \in E.$$

- **Matching**: is a set of edges without common vertices

$$\mathcal{M} \subseteq E \text{ such that for all } e_i, e_j \in \mathcal{M} \text{ with } e_i \neq e_j, e_i \cap e_j = \emptyset.$$

- **Connected component**: a maximal subgraph in which there is a path between every pair of nodes in the subgraph.

1.4 Node sets and cuts

For a set of nodes $S \subseteq V$ in a graph $G = (V, E)$, its complement set is denoted by.

$$\bar{S} = V \backslash S = \{i \in V : i \notin S\}$$

The *cut* of S is a measure of the weight/number of edges crossing between S and $\bar{S}$.

1.5 Discrete Optimization over Graphs

Many graph analysis problems amount to optimizing an objective function over a graph.

Example 1 ***Shortest path.*** *Given a source node $s \in V$ and target node $t \in V$, find the shortest path of edges between s and t.*

Example 2 ***Maximum bipartite matching.*** *Let $G = (V, A, B)$ be a bipartite graph. Find a matching $\mathcal{M}$ with maximum possible edge weight.*

Two related problems that will come up repeatedly in this class are **Minimum s-t cuts and maximum s-t flow.**

2 The Maximum s-t Flow Problem

Input to the Maximum s-t Flow Problem

- A weighted and directed graph $G = (V, E, w)$
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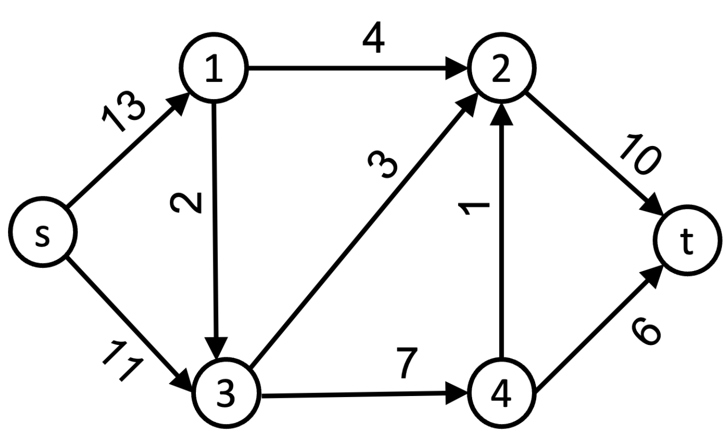
Goal: Route as much "flow" through the graph from s to t as possible, such that:

- The flow on an edge is bounded by its weight
- The flow into a node is equal to the flow out of a node (except for s and t)



One interpretation/application: transporting products/merchandise as efficiently as possible through a transportation network.

3 Defining s-t flows more formally

Given a weighted graph $G = (V, E, w)$, each $(u, v) \in E$ has a weight or *capacity* $w(u, v) = c(u, v)$.

A *flow* on G is a function

$$f: E \to \mathbb{R} \tag{1}$$

that satisfies two properties:

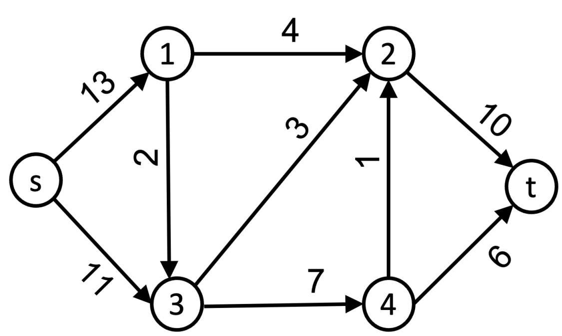
1. **Capacity constraints**: for each edge $(u, v) \in E$:

2. **Flow constraints**: for each node $v \notin \{s, t\}$

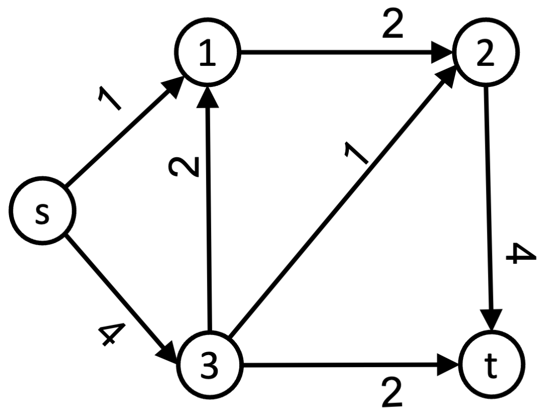
The *value* of a flow f is given by

$$|f| = \sum_{j\colon (s,j) \in E} f(s, j) - \sum_{u\colon (u,s) \in E} f(u, s) \tag{2}$$

Formal goal: find the flow function f^* with maximum value $|f^*|$.



Question 1 *What is the value of the flow f below? Is it a maximum flow?*



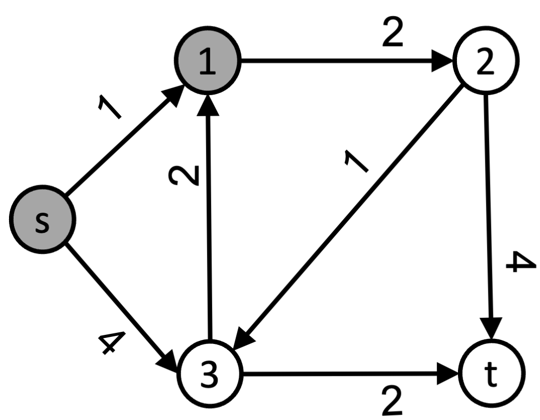
4 The minimum s-t cut problem

The minimum s-t cut problem takes the same type of input as the maximum s-t flow: a weighted directed graph $G = (V, E, w)$.

An *s-t cut* set is a set of nodes $S \subseteq V$ such that

The *value* of the cut is the weight of edges that cross from S to $V - S$. Formally:

What is the cut value below, where S is the set of gray nodes?



5 Relating minimum s-t cuts and maximum s-t flows

Lemma 3 *Let $G = (V, E, w)$ be a weighted directed graph. Let $S \subseteq V$ be any set with S be an s-t cut set, and let f be a flow. Then*