CSCE 689: Advanced Graph Algorithms

# Lecture 2: Maximum $s$-$t$ Flow

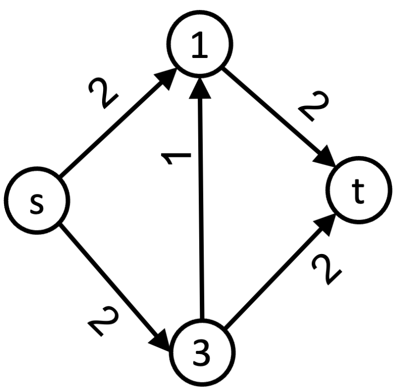
*Date: August 30* *Lecturer: Nate Veldt*

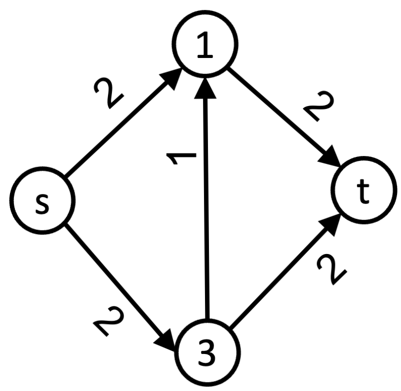
**Course Logistics**

- Homework 1 has been posted, due on Tuesday, Sept 13 at 11:59pm
- Intro video assignment posted
- Can now access recorded lectures via Canvas, course materials through veldt.engr.tamu.edu/689-fall22

## 1 Finding maximum $s$-$t$ flow

**Recap of our first idea.** Repeatedly find paths from $s$ to $t$, and keep adding flow until there are no more $s$-$t$ paths.




How do we correct this? Let's try to keep track of flow that we could "undo".

## 2 The Residual Graph

Given a flow $f$, for a pair of nodes $(u, v) \in V \times V$, the *residual capacity* for $(u, v)$ is

$$c_f(u, v) = c(u, v) - f(u, v) + f(v, u)$$

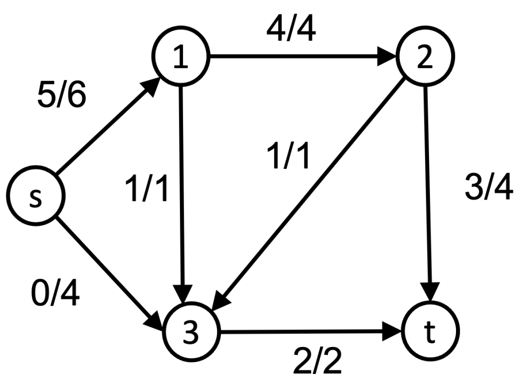
Informally, this is the amount of "space" left on the edge $c(u, v)$, plus the amount of flow from $v$ to $u$ that we could "undo".

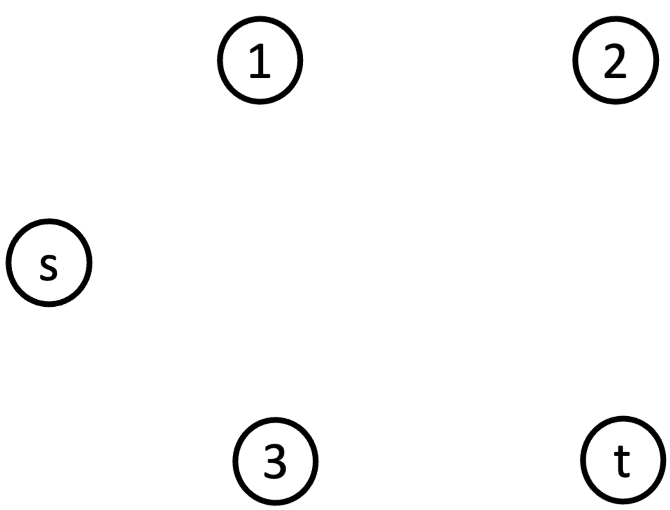
Given a flow $f$ for a graph $G = (V, E, w)$, the *residual graph* $G_f = (V, E_f)$ is the graph where the edge set

$$E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$$

This graph shows us where we can send more flow to improve on the flow $f$.

**Activity: draw the residual graph for the following flow**



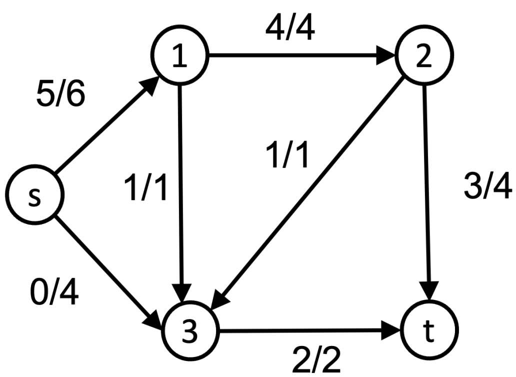
## 3 Augmenting Flows and Paths

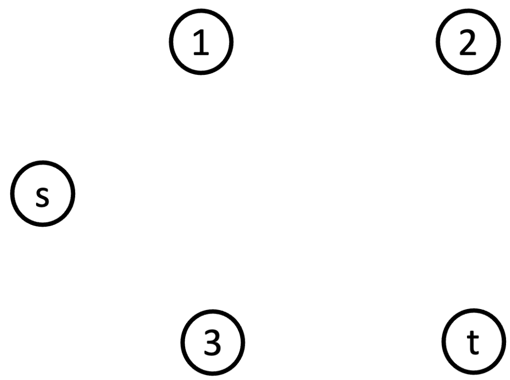
Let $f$ be an $s$-$t$ flow in $G = (V, E)$ and $f'$ be a flow in the residual graph $G_f = (V, E_f)$. Then we define the *augmentation* of $f$ by $f'$ as:

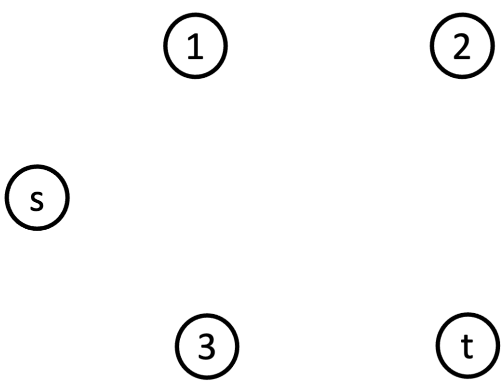
$$f \uparrow f' = f(u, v) + f'(u, v) - f'(v, u) \tag{1}$$

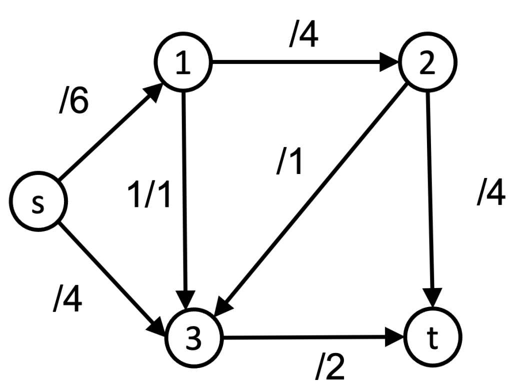
**Lemma 1.** *The function $f \uparrow f'$ is a valid flow in $G$, and it has flow value $|f| + |f'|$.*

Proof: a whole bunch of bookkeeping. We will skip this. But we can illustrate it below.



An *augmenting path* $p$ is a simple path (simple = no cycles) from $s$ to $t$ in the residual network $G_f$.

The *residual capacity* of this path $p$ is the maximum amount we can send on $p$:

$$c_f(p) = \min\{c_f(u, v) \colon (u, v) \text{ is in } p\}$$

Sending $c_f(p)$ flow along every edge in this path gives us a flow $f_p$ in $G_f$ that we can add to $f$ to improve it.

**Theorem 2.** *(Max-flow Min-cut Theorem) Let $f$ be an $s$-$t$ flow on some graph $G = (V, E)$. The following three conditions are equivalent:*

1. *$f$ is a maximum $s$-$t$ flow*
2. *There are no augmenting paths in the residual graph $G_f$*
3. *$|f| = \boldsymbol{cut}(S)$ for some $s$-$t$ cut set in $G$*

## 4 The Basic Ford-Fulkerson Algorithm

Idea: $f$ is a max-flow if and only if there are no augmenting paths. So let's just keep finding augmenting paths until we're done!

The Ford-Fulkerson algorithm will always maintain the invariant that for any pair $(u, v)$, at most one of $\{f(u, v), f(v, u)\}$ will be greater than zero.

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FordFulkersonBasic($G, s, t$)
  **for** $(u, v) \in E$ **do**
    $f(u, v) = 0$
  **while** there exists an $s$-$t$ path $p$ in $G_f$ **do**
    $c_f(p) = \min\{c_f(u, v) \colon (u, v) \text{ is in } p\}$
    **for** $(u, v) \in p$ **do**
      $m = \min\{c_f(p), f(v, u)\}$ // flow to "undo"
      $\ell = c_f(p) - m$ // new flow to send along $(u, v)$
      $f(v, u) \leftarrow f(v, u) - m$
      $f(u, v) \leftarrow f(u, v) + \ell$

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For $(u, v) \in p$, we first use any of the flow $c_f(p)$ to undo flow previously sent on $(v, u)$.

Then, if any of $c_f(p)$ remains, we send it along $(u, v)$.

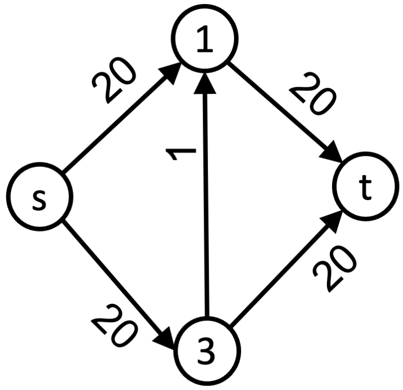
## 5 Runtime Analysis

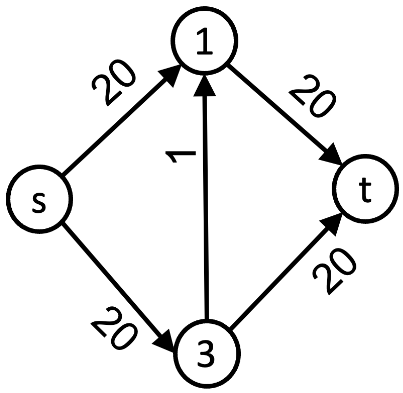
- $f^*$ is the maximum flow and $|f^*|$ the maximum flow value
- Assume all weights are integers.
- Let $f$ be the flow we are growing as the algorithm progresses.

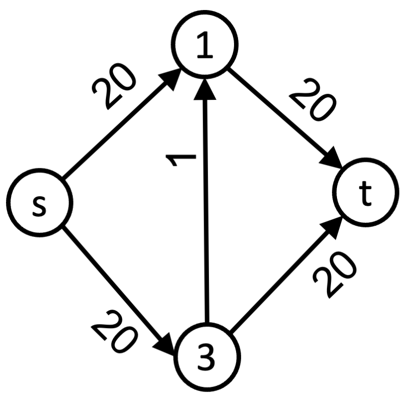
We need to answer the following questions:

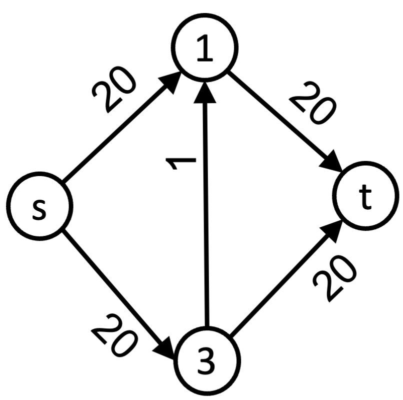
1. What is the runtime complexity for finding an $s$-$t$ path $p$ in $G_f$?
2. What is the minimum amount by which we can increase $f$ in each iteration?
3. What is the maximum number of paths we might have to find before we are done?
4. What is an overall runtime bound for FordFulkersonBasic?

## 6 How bad can the runtime be in practice?



# 7 The Edmonds-Karp Algorithm

The Edmonds-Karp Algorithm is a variation on Ford-Fulkerson that chooses an augmenting path $p$ by finding the directed path from $s$ to $t$ with the smallest number of edges. This is accomplished using a:

## 7.1 Shortest path distances increases monotonically

Let $f$ be an $s$-$t$ flow for input $G = (V, E, s, t)$ and $G_f$ be the residual graph. Define

$$\delta_f(s, v) = \text{ the shortest unweighted path distance from } s \text{ to } v \text{ in } G_f$$

**Lemma 3.** *For every $v \in V$, the distance $\delta_f(s, v)$ increases monotonically with each flow augmentation.*

Translation: as we keep finding augmenting paths $p$ and sending more flow $f_p$ to $f$, the distance between $s$ and every node either stays the same, or increases.

**Theorem 4.** *The total number of flow augmentation steps performed by Edmonds-Karp is $O(VE)$.*

*Proof.*
- Let $p$ be an augmenting path in $G_f$.

- An edge $(u, v) \in p$ is *critical* if $c_f(p) = c_f(u, v)$, meaning it is the smallest capacity edge in that path.

- When we push $c_f(p)$ flow through $p$, the edge $(u, v)$ disappears from $G_f$

- At least one edge on each path $p$ is critical.

- Claim: Each of the $|E|$ edges can be critical at most $|V|/2$ times.

**Proving the claim**: (u,v) becomes critical at most $|V|/2$ times.

- Let $u$ and $v$ be nodes in some edge in $E$.

- When $(u, v)$ is critical for the first time, $\delta_f(s, v) = \delta_f(s, u) + 1$

  Why?

- Then $(u, v)$ disappears from the residual graph, and can only re-appear after $(v, u)$ is on some future augmenting path. Say that $(v, u)$ is on an augmenting path when the new flow on $G$ is $f'$, then

$$\delta_{f'}(s, u) =$$

- We know that $\delta_f(s, v) \leq \delta_{f'}(s, v)$

- So we have

$$\delta_{f'}(s, u) =$$

- From the first to the second time $(u, v)$ becomes critical, the distance from $s$ to $u$ increases by at least 2.

- If $(u, v)$ becomes critical more than $|V|/2$ times, then the distance from $s$ to $u$ would be greater than $|V| - 2$.

- Thus, $(u, v)$ becomes critical at most $|V|/2 = O(V)$ times.

$\square$