1. Linear Algebra Review

1.1 Matrix concepts

$A \in \mathbb{R}^{m \times n}$ means A is a matrix with m rows and n columns; all entries are real-valued ($A_{ij} \in \mathbb{R}$)

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{12} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \\ A_{m1} & A_{m2} & & A_{mn} \end{bmatrix}$$

<u>Square</u>: $m = n$. Within square matrices, we have:

<u>Diagonal</u>: $A_{ij} = 0$ if $i \neq j$

e.g. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -12 \end{bmatrix}$

__Symmetric:__ $A_{ij} = A_{ji}$ for all $i,j \in [n]^2$

__invertible:__ Many ways to characterize invertibility of $A \in \mathbb{R}^{n \times n}$

- $\det(A) \neq 0$
- columns of A are linearly independent
- $\underset{n\times n}{A}\, \underset{n\times 1}{\underline{x}} = \underset{n\times 1}{\underline{0}}$ ← vector of zeros $\Rightarrow$ $\underline{x} = \underline{0}$ is the only solution

$\vdots$

1.2 Vector Concepts

$\underline{x} \in \mathbb{R}^{n \times 1}$ is a real-valued vector

$\|\underline{x}\|_2 = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2}$ is the norm of $\underline{x}$, where x_i is the ith entry

$$\underline{x}^T = [x_1 \ x_2 \ \cdots \ x_n]^T$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(we'll view vectors as column vectors in this course)

Def. Two nonzero vectors $\underline{x}, \underline{y}$ are orthogonal if $\underline{x}^T \underline{y} = 0$.

Def. A vector is nonzero if at least 1 entry is nonzero, e.g. $x = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$.

Def. $\underline{x}$ is a unit vector if $\|x\|_2 = 1$.

1.3 Linear systems

Let $A \in \mathbb{R}^{n \times m}$, $\underline{b} \in \mathbb{R}^m$. A linear system of equations is a system of equations of the form

$$A \underline{x} = \underline{b}$$

where $\underline{x}$ is a vector of unknown variables.

Important note: When A is square and invertible, the solution to $A\underline{x} = \underline{b}$ is unique and equals

$$\underline{x} = A^{-1} \underline{b}$$

However, in practice you should NOT compute A^{-1} explicitly, as other ways to solve the system are faster and more accurate (recall Gaussian Elimination; many other methods as well).

1.4 Eigenvalues and eigenvectors

Def. Let $A \in \mathbb{R}^{n \times n}$, and assume

$$A\underline{x} = \lambda \underline{x}$$

for a nonzero vector $\underline{x}$ and scalar $\lambda \in \mathbb{C}$. Then λ is an eigenvalue of A and x is a corresponding eigenvector.

Review: If $A\underline{x} = \lambda \underline{x}$, then

$$(A - \lambda I)\underline{x} = \underline{0}$$

Since x is nonzero, $(A - \lambda I)$ is not invertible. The eigenvalues are exactly the solutions to the equation

$$\det(A-\lambda I)=0,$$

where $\det(A-\lambda I)$ is a polynomial in terms of λ. (the characteristic poly of A)

E.g. $A=\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow A-\lambda I=\begin{bmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$

$$\det(A-\lambda I)=(2-\lambda)(1-\lambda)-0\cdot 1=(2-\lambda)(1-\lambda)$$

$\Rightarrow$ eigenvalues are $\lambda=1$, $\lambda=2$.

Fact: Since every polynomial of degree n has n roots, counting multiplicity, every matrix $A\in\mathbb{R}^{n\times n}$ has n eigenvalues (counting multiplicity).

Fact: If A is symmetric, all of its eigenvalues are real