1. Linear Algebra Review

1.1 Matrix concepts

$A \in \mathbb{R}^{m \times n}$ means A is a matrix with m rows and n columns; all entries are real-valued ($A_{ij} \in \mathbb{R}$)

A_{ij}

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{12} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \\ A_{m1} & A_{m2} & & A_{mn} \end{bmatrix}$$

<u>Square</u>: $m = n$. Within square matrices, we have:

<u>Diagonal</u>: $A_{ij} = 0$ if $i \neq j$

e.g. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 12 \end{bmatrix}$

Symmetric: $\underline{A_{ij} = A_{ji}}$ for all $i,j \in [n]^2$

invertible: Many ways to characterize invertibility of $\underline{A \in \mathbb{R}^{n \times n}}$

- $\det(A) \neq 0$
- columns of A are linearly independent
- $\underset{n\times n}{A}\,\underset{n\times 1}{\underline{x}} = \underset{n\times 1}{\underline{0}}$ (vector of zeros) $\Rightarrow$ $\underline{x} = \underline{0}$ is the only solution

$\vdots$

1.2 Vector Concepts

$\underline{x} \in \mathbb{R}^{n \times 1}$ is a real-valued vector

$\underline{\|\underline{x}\|_2} = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2}$ is the norm of $\underline{x}$, where x_i is the ith entry

$$\underline{x}^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$$

$$\boxed{\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}$$

(we'll view vectors as column vectors in this course)

Def. Two nonzero vectors $\underline{x}, \underline{y}$ are orthogonal if $\underline{x}^T \underline{y} = 0$.

Def. A vector is nonzero if at least 1 entry is nonzero, e.g. $x = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$.

Def. $\underline{x}$ is a unit vector if $\|x\|_2 = 1$.

1.3 Linear systems

Let $A \in \mathbb{R}^{n\times m}$, $\underline{b} \in \mathbb{R}^{m}$. A linear system of equations is a system of equations of the form

$$\overset{n\times n}{A}\,\underline{x} = \underline{b}$$

where $\underline{x}$ is a vector of unknown variables.

Important note: When A is square and invertible, the solution to $A\underline{x} = \underline{b}$ is unique and equals

$$\underline{x} = A^{-1}\underline{b}$$

However, in practice you should NOT compute A^{-1} explicitly, as other ways to solve the system are faster and more accurate (recall Gaussian Elimination; many other methods as well).

1.4 Eigenvalues and eigenvectors

Def. Let $A \in \mathbb{R}^{n\times n}$, and assume

$$\boxed{A\underline{x} = \lambda \underline{x}}$$

for a nonzero vector $\underline{x}$ and scalar $\lambda \in \mathbb{C}$. Then λ is an eigenvalue of A and x is a corresponding eigenvector.

Review: If $A\underline{x} = \lambda \underline{x}$, then

$$\boxed{(A - \lambda I)}\,\underline{x} = \underline{0}$$

Since x is nonzero, $(A - \lambda I)$ is not invertible. The eigenvalues are exactly the solutions to the equation

$$\boxed{\det(A-\lambda I)=0,}$$

where $\det(A-\lambda I)$ is a polynomial in terms of λ. (the characteristic poly of A)

E.g. $A=\begin{bmatrix}2&1\\0&1\end{bmatrix} \Rightarrow A-\lambda I=\begin{bmatrix}2-\lambda&1\\0&1-\lambda\end{bmatrix}$

$$\det(A-\lambda I)=(2-\lambda)(1-\lambda)-0\cdot 1=(2-\lambda)(1-\lambda)$$

$\Rightarrow$ eigenvalues are $\lambda=1,\ \lambda=2$.

Fact: Since every polynomial of degree n has n roots, counting multiplicity, every matrix $A\in\mathbb{R}^{n\times n}$ has n eigenvalues (counting multiplicity).

$\boxed{\text{Fact}}$: If A is symmetric, all of its eigenvalues are real

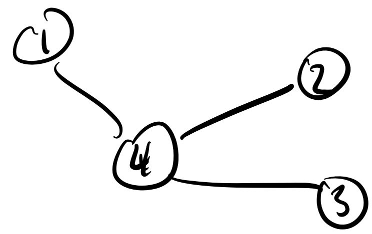
2. Encoding Graphs as Matrices

Let $G=(V,E)$ be a simple graph with vertex set $V=\{1,2,\dots,n\}$

Adjacency: The adjacency matrix $A \in \{0,1\}^{n\times n}$ is defined by setting

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Degree Matrix



$d_i = 3$

$= |N(i)|$

D is the diagonal matrix where $D_{ii} = d_i$

(undirected unweighted)

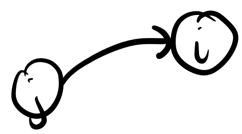
$$D = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 3 \end{bmatrix}$$

Laplacian Matrix

$L = D - A$ (undirected unweighted)

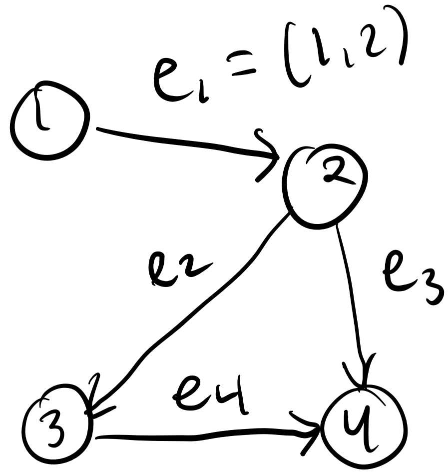
Incidence Matrix

$B \in \{-1, 1, 0\}^{|V| \times |E|}$ is defined as follows for a unweighted directed graph



Incidence

$$B_{i,e} = \begin{cases} 1 & \text{if } e = (j, i) \\ -1 & \text{if } e = (i, j) \\ 0 & \text{otherwise} \end{cases}$$



$$B = \begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ \hline 1 & -1 & 0 & 0 & 0 \\ 2 & 1 & -1 & -1 & 0 \\ 3 & 0 & 1 & 0 & -1 \\ 4 & 0 & 0 & 1 & 1 \end{array}$$

Unsigned incidence matrix

(unweighted undirected)

$\hat{B}$ is defined by

$$\hat{B}_{ij} = |B_{ij}|$$

$$\hat{B} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Fact: $\underline{L} = \underset{|V|\times|E|}{\underline{B}}\ \underset{|E|\times|V|}{\underline{B}^T}$

3. Encoding graph concepts as linear algebra

Let $A \in \{0,1\}^{n \times n}$ be the adjacency matrix for a simple graph $G = (V, E)$ $\quad n = |V|$

$V = \{1, 2, \ldots, n\}$

Let

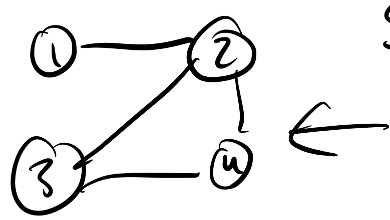
$$\underline{e} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{n \times 1}$$

For $S \subseteq V$ let $\underline{e}_S \in \mathbb{R}^{n \times 1}$ be defined so that

$$\underline{e}_S(i) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

If $S = \{i\}$ we write $e_{\{i\}} = e_i$.

e.g.



$S = \{3, 4\}$

$$e_S = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

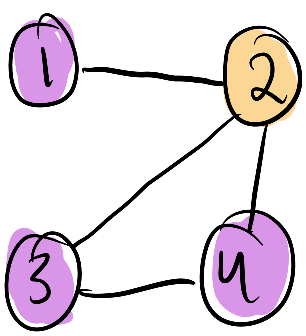
3.1 Node neighborhoods

If $\underline{x} = A\,\underline{e}_i$ then

$x = \underline{e}_S$ where $S = N(i)$

e.g.

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{e_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$



In general if $\underline{x} = A^t \underline{e}_i$ then

$\underline{x}(j) > 0$ if j is within

t hops of node i.

3.2 Hops and paths

$t \in \mathbb{N}$

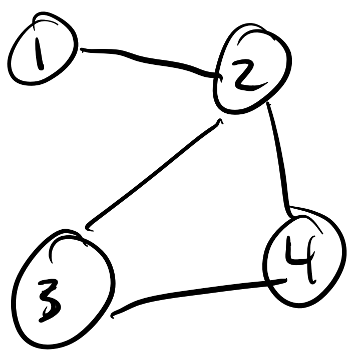
Fact: A^t_{ij} = the number of paths of length t frome node i to node j.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \boxed{1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{A e_i = x}$$

$$\underline{A^2 e_i} = A \cdot A e_i = A x$$

$$A^2 e_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \overbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}}^{Ae_i} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$



Corollary: $\boxed{\dfrac{\operatorname{trace}(A^3)}{6}}$ = number of triangles in the graph

(HW)

Fact: $e_S^T L e_S = \operatorname{cut}(S)$

$S \subseteq V$

(HW)

4. Linear Programming

A linear program is a linear optimization problem involving linear constraints.

The standard form of a linear program (LP) is given by

given

$$\underset{x}{\underline{\min}} \sum_{j=1}^{n} c_j x_j$$

given

$$\text{subject to} \quad \underline{\sum_{j=1}^{n} A_{ij} x_j = b_j} \quad \text{for } i=1,2,\dots,m$$

$$\underline{x_j \geq 0} \quad \text{for } j=1,2,\dots,n$$

In matrix form:

$$\begin{aligned} \min \quad & \underline{c}^T \underline{x} \\ \text{s.t.} \quad & A\underline{x} = \underline{b} \\ & \underline{x} \geq 0 \end{aligned} \qquad (1)$$

Example: $\min \quad x_1 + x_2$

s.t. $\quad 6x_1 + x_2 = 3$

$$x_1 \geq 0$$
$$x_2 \geq 0$$

There are many different variations on standard form. For any LP, we can convert to standard form.

Variation		**Conversion**
$\max \; c^T x$	$\longleftrightarrow$	$\min \; \hat{c}^T x$ $\hat{c} = -c$
$x_1 + x_2 \leq 3$	$\longleftrightarrow$	$x_1 + x_2 + s = 3$ $s \geq 0$

$$x_1 + x_2 = 3 \quad \longleftrightarrow \quad \begin{aligned} x_1 + x_2 &\leq 3 \\ x_1 + x_2 &\geq 3 \\ (-x_1 - x_2 &\leq -3) \end{aligned}$$

$$x_1 \text{ unrestricted} \quad \longleftrightarrow \quad \begin{aligned} &x_1^+ \geq 0 \quad x_1^- \geq 0 \\ &x_1 = x_1^+ - x_1^- \\ &(\text{replace } x_1) \end{aligned}$$

You can also write every LP in <u>canonical form</u>:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

<u>Some definitions</u>: Given (1)

- $\underline{x}$ is feasible if $A\underline{x} = \underline{b}$
- (1) is feasible if it has at least one feasible $\underline{x}$, otherwise "infeasible"

• $\underline{x}^*$ is optimal if it is feasible and $\underline{c}^T\underline{x}^* \leq \underline{c}^T\underline{x}$ for all feasible $\underline{x}$

• The LP is unbounded if $\exists\, p \in \mathbb{R}$ such that for all $q \leq p$ there is a feasible $\underline{x}$ with $c^T\underline{x} \leq q$

<u>Example</u>: s-t flow ↙

$$\max \sum_{u:(s,u)\in E} f_{su} - \sum_{v:(v,s)\in E} f_{vs}$$

for each $(u,v)\in E$ $\quad f_{uv} \leq c(u,v)$

for $v \in V - \{s,t\}$

$$\sum_{(u,v)\in E} f_{uv} = \sum_{(v,u)\in E} f_{vu}$$

$f_{uv} \geq 0$ for all $(u,v)\in E$

Binary and Integer Linear Programs

In a binary linear program, the variables are restricted to being in $\{0,1\}$.

In an integer LP the variables are in $\mathbb{Z}$.

In general, these problems are NP-hard to solve.

could be in $\mathbb{R}$ (might not be integers)

$$\max \quad \underline{c}^T \underline{x}$$
$$A\underline{x} \geq \underline{b}$$
$$\underline{x} \in \mathbb{Z}^n$$