CSCE 689: Advanced Graph Algorithms

Lecture 6: Vertex Cover, Matching, and LP duality

Date: Sept 13, 2022 *Lecturer: Nate Veldt*

Course logistics

- HW 1 due end of today.
- Solution key will be posted by midnight. You are allowed to submit 1 day late with a 15% late penalty.

Recap: Last week

- LP duality
- Totally unimodular matrices in linear programming

This week: Vertex Cover, Matchings, and More LP duality

Recap: LP duality

Let $\underline{c} \in \mathbb{Q}^n$, $\underline{b} \in \mathbb{Q}^m$, $A \in \mathbb{Q}^{m \times n}$, and consider the linear program (LP) in canonical form:

$$\begin{aligned} \min_{\underline{x} \in \mathbb{Q}^n} \quad & \underline{c}^T \underline{x} \\ \text{subject to} \quad & \boxed{A\underline{x} \geq \underline{b}} \\ & \underline{x} \geq 0 \end{aligned} \qquad \begin{array}{c} (1) \\ \text{(primal)} \end{array}$$

The <u>dual</u> of LP (1) is

$$\max_{y \in \mathbb{Q}^m} \quad \underline{b}^T \underline{y}$$

$$\boxed{A^T \underline{y} \leq \underline{c}} \qquad (2)$$

$$\underline{y} \geq 0 \qquad \text{(dual)}$$

Theorem (weak duality): Let $\underline{x}$ be feasible for (1) and $\underline{y}$ is feasible for (2), then

$$\underline{c}^T \underline{x} \geq \underline{b}^T \underline{y}$$

Proof: $\boxed{\underline{c}^T} \underline{x} \geq \boxed{y^T A} \underline{x} \geq y^T \boxed{\underline{b}}$

Theorem (strong duality): Let $(\underline{x}^*, \underline{y}^*)$ be optimal for LP (1) and LP (2) respectively, then

$$\underline{c}^T \underline{x}^* = \underline{b}^T \underline{y}^*$$

(assuming both LPs are feasible)

Find the dual

Strategy 1: learn a set of rules and apply them

(1 variable for each constraint, 1 constraint for each variable)

Strategy 2: convert the LP into standard or canonical form and apply one of the following patterns

	Primal $\longleftrightarrow$	Duals
canonical	$\min c^T x$ $Ax \geq b$ $x \geq 0$	$\max b^T y$ $A^T y \leq c$ $y \geq 0$
standard	$\min c^T x$ $Ax = b$ $x \geq 0$	$\max b^T y$ $A^T y \leq c$

Common examples in graph analysis

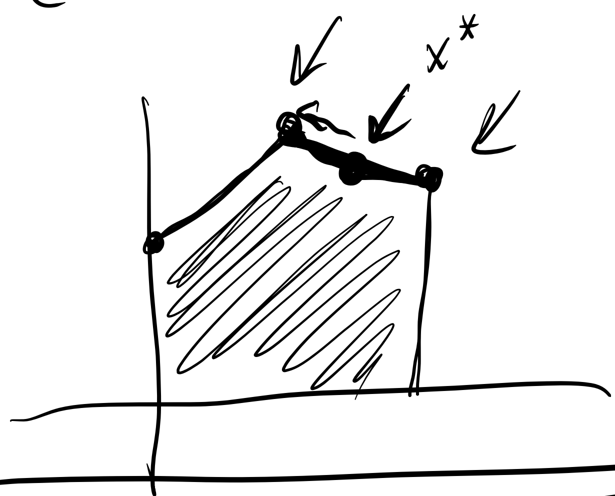
- The dual of the max s-t flow LP is the min s-t cut LP
- In bipartite graphs the dual of the max matching problem is the min vertex cover problem.

TU Matrices in Linear Programming

For the LP

$$\boxed{\begin{aligned} \min \quad & c^T x \\ & Ax \geq b \\ & x \geq 0 \end{aligned}} \qquad \begin{aligned} &(1) \\ &x \in \mathbb{Z}^n \end{aligned}$$

Theorem TU: If $b \in \mathbb{Z}^m$ and A is totally unimodular (all square submatrices have determinant 0, 1, or -1) then there exists $x^* \in \mathbb{Z}^n$ that is optimal for (1).



Vertex Cover and Matchings

Def: a matching $M \subseteq E$ of a simple graph $G=(V,E)$ is a vertex disjoint set of edges.

Def: a vertex cover $C \subseteq V$ of $G=(V,E)$ is a set of nodes such that every $e \in E$ is adjacent to at least one node in C.

i.e. $|e \cap C| \geq 0 \quad \forall \; e \in E.$

<u>Lemma</u>: Let $G = (V, E)$ be a simple graph, C a vertex cover, and M be a matching then

$$|M| \leq |C|$$

LP duality for Vertex Cover

The minimum vertex cover problem (Min-VC) is encoded by the binary linear program (BLP)

$$\min \sum_{v \in V} x_v \qquad (1)$$

$$\text{s.t.} \quad x_u + x_v \geq 1 \quad \forall \; (u,v) \in E$$

$$\boxed{x_u \in \{0,1\}} \quad \forall \; u \in V$$

Min-VC is NP-hard.

Let BLP-VC be the optimal solution value.

The "LP relaxation" of this BLP is

(2)

$$\begin{aligned} \min \quad & \sum_{v \in V} x_v \cdot 1 & \\ \text{s.t.} \quad & x_u + x_v \geq 1 & \forall\ (u,v) \in E \\ & 0 \leq x_u \cancel{\leq 1} & \forall\ u \in V \end{aligned}$$

Let LP-VC be the optimal solution value.

Observation: LP-VC $\boxed{\substack{\leq \\ \geq}}$ BLP-VC
?

The dual of LP (2) is given by:

(3) $$\boxed{\begin{aligned} \max \quad & \sum_{e \in E} y_e \cdot 1 \\ \text{s.t.} \quad \forall\, u \in V \quad & \sum_{e:\, u \in e} y_e \le 1 \\ & y_e \geq 0 \end{aligned}}$$

Let LP-M be the optimal solution value. Notice LP-M = LP-VC.

Consider the BLP obtained by forcing $y_e \in \{0,1\}$ instead of $y_e \geq 0$:

$$\max \sum_{e \in E} y_e$$

$$\text{s.t. } \forall\, u \in V \quad \sum_{e:\, u \in e} y_e \le 1$$

$$y_e \in \{0,1\}.$$

Let BLP-M be the optimal solution value. Observe:

$$\text{BLP-M} \; \boxed{\begin{matrix}\le \\ \ge\end{matrix}} \; \text{LP-M}$$

?

$$\text{BLP-M} \le \text{LP-M} = \text{LP-VC} \le \text{BLP-VC}$$

Lemma: M and C are a matching and a cover, then

$$|M| \le |C|$$

Theorem: If $G=(V,E)$ is a bipartite graph and M^* and C^* are optimal matching and cover, then $|M^*| = |C^*|$.

Proof: The constraint matrix for LP (3) is $\hat{B}$, the unsigned incidence matrix, which is TU. By Theorem TU,

$$|M^*| = BLP\text{-}M = LP\text{-}M = LP\text{-}VC = BLP\text{-}VC = |C^*|$$

(application of LP duality and TU)